

The second largest Erdős-Ko-Rado sets of generators of the hyperbolic quadrics $\mathcal{Q}^+(4n+1, q)$

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Abstract

An Erdős-Ko-Rado set of generators of a hyperbolic quadric is a set of generators which are pairwise not disjoint. In this article we classify the second largest maximal Erdős-Ko-Rado set of generators of the hyperbolic quadrics $\mathcal{Q}^+(4n+1, q)$, $q \geq 3$.

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1 Introduction

1.1 Hyperbolic quadrics

Finite classical polar spaces are incidence structures consisting of the subspaces of a vector space over a finite field that are totally isotropic with respect to a certain non-degenerate sesquilinear or quadratic form. Thereby incidence is defined by the inclusion relation. Their maximal totally isotropic subspaces are called generators. Since the geometry of the subspaces of a vector space is a projective space, these classical polar spaces can be considered as substructures of a projective space. The projective space of dimension n over the finite field \mathbb{F}_q will be denoted by $\text{PG}(n, q)$. Throughout this paper we will use the projective dimension for subspaces of polar spaces: a point is 0-dimensional, a line is 1-dimensional, ...

If such a classical polar space arises from a non-degenerate quadratic form it is called a quadric. As a substructure of the projective space it is defined by a non-degenerate homogeneous quadratic equation of degree 2. According to this quadratic form (or quadratic equation), three types of quadrics are distinguished: elliptic quadrics, parabolic quadrics and hyperbolic quadrics. The quadrics in $\text{PG}(2m+1, q)$ which can be defined by the standard equation

$$X_0X_1 + X_2X_3 + \cdots + X_{2m}X_{2m+1} = 0,$$

after projective transformation, are the hyperbolic quadrics. A hyperbolic quadric in $\text{PG}(2m+1, q)$ is denoted by $\mathcal{Q}^+(2m+1, q)$.

An extensive introduction to quadrics can be found in [8, Chapter 22]. We list some basic facts which will be used in this article without reference.

Theorem 1.1. Let $\mathcal{Q}^+(2m+1, q)$ be a hyperbolic quadric in $\text{PG}(2m+1, q)$ and let Ω be its set of generators.

- The generators of $\mathcal{Q}^+(2m+1, q)$ are m -spaces.
- The number of elements of Ω is $\prod_{i=0}^m (q^i + 1)$.
- The equivalence relation \sim on Ω , defined by $\pi \sim \pi' \Leftrightarrow \dim(\pi \cap \pi') = m \pmod{2}$ for any $\pi, \pi' \in \Omega$, partitions Ω in two generator classes of the same size, commonly called the Latin and the Greek generators.

- Let π_i be an i -dimensional subspace of $\mathcal{Q}^+(2m+1, q)$, $0 \leq i \leq m$. The tangent space $T_{\pi_i}(\mathcal{Q}^+(2m+1, q))$ to $\mathcal{Q}^+(2m+1, q)$ in π_i is a $(2m-i)$ -space in $\text{PG}(2m+1, q)$. The intersection $T_{\pi_i}(\mathcal{Q}^+(2m+1, q)) \cap \mathcal{Q}^+(2m+1, q)$ is a cone with vertex π_i and base a hyperbolic quadric $\mathcal{Q}^+(2(m-i)-1, q)$.

To simplify some of the notations in this article, we recall the definition of the Gaussian binomial coefficient:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{i=1}^k \left(\frac{q^{n+1-i} - 1}{q^i - 1} \right).$$

The number of k -spaces in $\text{PG}(n, q)$ is given by $\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q$.

1.2 Erdős-Ko-Rado sets

The original Erdős-Ko-Rado theorem was stated in [5].

Theorem 1.2 ([5, Theorem 1]). If \mathcal{S} is a family of subsets of size k in a set Ω with $|\Omega| = n$ and $n \geq 2k$, such that the elements of \mathcal{S} are pairwise not disjoint, then $|\mathcal{S}| \leq \binom{n-1}{k-1}$.

In honour of this result, a family of subsets of fixed size in a set, pairwise not disjoint, is called an Erdős-Ko-Rado set. Note that a point-pencil, a family of subsets of size k through a fixed element, meets the upper bound of the previous theorem. It was proved that this is the unique example meeting this upper bound if $n \geq 2k+1$. This is part of the following result which also classifies the second largest Erdős-Ko-Rado set.

Theorem 1.3 ([7]). Let Ω be a set of size n and let \mathcal{S} be an Erdős-Ko-Rado set of k -subsets in Ω , $k \geq 3$ and $n \geq 2k+1$. If there is no element in Ω which is contained in all subsets in \mathcal{S} , then

$$|\mathcal{S}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

Moreover, equality holds if and only if

- either \mathcal{S} is the union of $\{F\}$, for some fixed k -subset F , and the set of all k -subsets G of Ω containing a fixed element $x \notin F$, such that $G \cap F \neq \emptyset$,
- or else $k = 3$ and \mathcal{S} is the set of all subsets of size 3 having an intersection of size at least 2 with a fixed subset F of size 3.

This problem has been generalised to different geometrical structures, such as vector spaces, polar spaces and designs. We refer to the survey articles [1, 4] and for detailed reading to [6, 13, 14, 15] among others.

In this article we will study the Erdős-Ko-Rado problem for generators of hyperbolic quadrics.

Definition 1.4. Let $\mathcal{Q}^+(2m+1, q)$ be a hyperbolic quadric and let Ω be its set of generators. An Erdős-Ko-Rado set \mathcal{S} of generators of $\mathcal{Q}^+(2m+1, q)$ is a subset of Ω such that any two elements of \mathcal{S} have a nonempty intersection. If \mathcal{S} is not extendable regarding this condition it is called maximal.

The Erdős-Ko-Rado problem now asks for the classification of the large maximal Erdős-Ko-Rado sets.

For a hyperbolic quadric $\mathcal{Q}^+(2m+1, q)$, its Erdős-Ko-Rado sets heavily depend on the parity of m . If m is even, any two generators belonging to the same class meet in at least a point; if m is odd, any two generators belonging to different classes meet in at least a point. In the latter case, it is therefore sufficient to study Erdős-Ko-Rado sets of generators of one class. In this article we study the former case. We give an example of a maximal Erdős-Ko-Rado set.

Example 1.5. Let $\mathcal{Q}^+(4n+1, q)$ be a hyperbolic quadric and let \mathcal{S} be the set of generators of one class. Since any two generators belonging to the same class cannot be disjoint, \mathcal{S} is an Erdős-Ko-Rado set. Moreover, it is maximal since no generator of the other class can meet all generators of the class corresponding to \mathcal{S} .

Note that $|\mathcal{S}| = \prod_{i=1}^{2n} (q^i + 1)$, half of the total number of generators.

This is an important example, as the next theorem shows.

Theorem 1.6 ([13, Theorem 9 and Theorem 16]). Let \mathcal{S} be an Erdős-Ko-Rado set of generators of $\mathcal{Q}^+(4n+1, q)$. Then, $|\mathcal{S}| \leq \prod_{i=1}^{2n} (q^i + 1)$. Furthermore, if $|\mathcal{S}| = \prod_{i=1}^{2n} (q^i + 1)$, then \mathcal{S} is the Erdős-Ko-Rado set described in Example 1.5.

So, for general hyperbolic quadrics $\mathcal{Q}^+(4n+1, q)$ the largest Erdős-Ko-Rado sets of generators have been classified. In fact, in [13], the largest Erdős-Ko-Rado sets of generators were classified for all finite classical polar spaces except the Hermitian varieties $\mathcal{H}(4n+1, q^2)$, $n \geq 2$. For those, the best result was obtained in [9].

We return to hyperbolic quadrics. For the smallest nontrivial case, a complete classification of Erdős-Ko-Rado sets of generators is known.

Theorem 1.7 ([2, Theorem 3.5]). Let \mathcal{S} be a maximal Erdős-Ko-Rado set of generators of $\mathcal{Q}^+(5, q)$. Then, \mathcal{S} is the set of all generators of one class, the set of generators meeting a fixed generator in at least a line, or the set of all generators through a fixed point.

These examples contain $q^3 + q^2 + q + 1$, $q^2 + q + 2$ and $2q + 2$ generators, respectively. We note that the proof of this theorem in [2] relies on the dual polar graph $D_3(q)$. This is the graph whose vertices are the generators of $\mathcal{Q}^+(5, q)$; two vertices are adjacent if and only the corresponding generators meet in a line (in general, in a hyperplane of a generator). The authors of [2] study the maximal $\{0, 1, 2\}$ -cliques in this graph (and other dual polar graphs); these are vertex sets such that any two vertices in the set are at distance 0, 1 or 2 in the graph. So, a $\{0, 1, 2\}$ -clique in $D_3(q)$ corresponds to a set of generators (planes) of $\mathcal{Q}^+(5, q)$ that pairwise meet in a plane, line or point, i.e. an Erdős-Ko-Rado set.

Since the largest Erdős-Ko-Rado sets of generators have been classified, we focus on other maximal Erdős-Ko-Rado sets of generators. In this article we classify the second largest maximal Erdős-Ko-Rado sets of generators of $\mathcal{Q}^+(4n+1, q)$. This result can be found in Theorem 3.7. This is an analogue of Theorem 1.3.

2 Counting results

We recall two counting results, one about subspaces and one about generators.

Theorem 2.1 ([12, Section 170]). The number of j -spaces skew to a fixed k -space in $\text{PG}(n, q)$ equals $q^{(k+1)(j+1)} \begin{bmatrix} n-k \\ j+1 \end{bmatrix}_q$.

Theorem 2.2 ([10, Corollary 5]). Let $\mathcal{Q}^+(2m+1, q)$ be a hyperbolic quadric, and let π_1 and π_2 be two generators of $\mathcal{Q}^+(2m+1, q)$ meeting in a j -dimensional space. The number of generators skew to both π_1 and π_2 equals

$$b_j^m := \begin{cases} q^{2\binom{(m+j)/2+1}{2} - \binom{j+1}{2}} \prod_{i=1}^{(m-j)/2} (q^{2i-1} - 1) & m \equiv j \pmod{2} \\ 0 & m \equiv j+1 \pmod{2} \end{cases}.$$

Now, we present a new counting result.

Lemma 2.3. Let $\mathcal{Q}^+(4n+1, q)$ be a hyperbolic quadric and let π_1 and π_2 be two generators of the same class on $\mathcal{Q}^+(4n+1, q)$ meeting in a j -dimensional space, $0 \leq j \leq 2n$ and j even. The

number of generators meeting π_1 , but not π_2 , equals

$$v_j^n := \sum_{i=\frac{j}{2}}^{n-1} q^{(2n-2i)(j+1)} \begin{bmatrix} 2n-j \\ 2n-2i \end{bmatrix}_q b_j^{2i}.$$

Proof. All generators belonging to the same class as π_1 and π_2 , meet both, hence cannot meet precisely one of them. Let π be a generator of the other class that meets π_1 . The intersection $\tau = \pi_1 \cap \pi$ is a $(2n-2i-1)$ -space, for an i fulfilling $\frac{j}{2} \leq i \leq n-1$. Let $\bar{\tau}$ be the tangent space in τ to $\mathcal{Q}^+(4n+1, q)$; it is $(2n+2i+1)$ -dimensional. The tangent space $\bar{\tau}$ contains π_1 and meets π_2 in a $(2i)$ -space through $\pi_1 \cap \pi_2$. The intersection $\bar{\tau} \cap \mathcal{Q}^+(4n+1, q)$ is a cone with vertex τ and base a hyperbolic quadric $\mathcal{Q}^+(4i+1, q)$. We can choose the ambient space σ of this base $\mathcal{Q}^+(4i+1, q)$ to contain $\bar{\tau} \cap \pi_2$.

Any generator through τ now corresponds to a generator of this base quadric $\mathcal{Q}^+(4i+1, q)$. Moreover, a generator through τ meeting π_1 in τ and disjoint to π_2 corresponds to a generator of $\mathcal{Q}^+(4i+1, q)$ skew to both $\bar{\tau} \cap \pi_2 = \sigma \cap \pi_2$ and $\sigma \cap \pi_1$, which both are generators of the base quadric. Since $(\sigma \cap \pi_1) \cap (\sigma \cap \pi_2) = \pi_1 \cap \pi_2$, the number of such generators equals b_j^{2i} .

So, the total number of generators meeting π_1 , but not π_2 , equals

$$\sum_{i=\frac{j}{2}}^{n-1} q^{(2n-2i)(j+1)} \begin{bmatrix} 2n-j \\ 2n-2i \end{bmatrix}_q b_j^{2i}.$$

Here we used the result from Theorem 2.1 to count the number of $(2n-2i-1)$ -spaces in π_1 that are skew to $\pi_1 \cap \pi_2$. \square

Corollary 2.4. Let $\mathcal{Q}^+(4n+1, q)$ be a hyperbolic quadric and let π_1 and π_2 be two generators of the same class on $\mathcal{Q}^+(4n+1, q)$ meeting in a $2(n-t)$ -dimensional space, $0 \leq t \leq n$. The number of generators not meeting both π_1 and π_2 equals

$$W_t^n(q) := q^{2n^2+n-t^2} \left(\prod_{k=1}^t (q^{2k-1} - 1) + 2 \sum_{i=1}^t \begin{bmatrix} 2t \\ 2i \end{bmatrix}_q q^{i^2+i-2it} \prod_{k=1}^{t-i} (q^{2k-1} - 1) \right).$$

Proof. Using the notations from Theorem 2.2 and Lemma 2.3, we find that the number of generators not meeting both π_1 and π_2 equals $b_{2(n-t)}^{2n} + 2v_{2(n-t)}^n$. Using the results from Theorem 2.2 and Lemma 2.3, we find that

$$\begin{aligned} W_t^n(q) &= b_{2(n-t)}^{2n} + 2v_{2(n-t)}^n \\ &= q^{2n^2+n-t^2} \prod_{k=1}^t (q^{2k-1} - 1) + 2 \sum_{i=n-t}^{n-1} q^{(2n-2i)(2n-2t+1)} \begin{bmatrix} 2t \\ 2n-2i \end{bmatrix}_q b_{2(n-t)}^{2i} \\ &= q^{2n^2+n-t^2} \prod_{k=1}^t (q^{2k-1} - 1) + 2 \sum_{i=1}^t q^{2i(2n-2t+1)} \begin{bmatrix} 2t \\ 2i \end{bmatrix}_q b_{2(n-t)}^{2(n-i)} \\ &= q^{2n^2+n-t^2} \prod_{k=1}^t (q^{2k-1} - 1) + 2 \sum_{i=1}^t q^{2i(2n-2t+1)} \begin{bmatrix} 2t \\ 2i \end{bmatrix}_q q^{2(n-i)^2+(n-i)-(t-i)^2} \prod_{k=1}^{t-i} (q^{2k-1} - 1) \\ &= q^{2n^2+n-t^2} \left(\prod_{k=1}^t (q^{2k-1} - 1) + 2 \sum_{i=1}^t \begin{bmatrix} 2t \\ 2i \end{bmatrix}_q q^{i^2+i-2it} \prod_{k=1}^{t-i} (q^{2k-1} - 1) \right). \end{aligned} \quad \square$$

3 Classification results

Example 3.1. Let π be a generator of the hyperbolic quadric $\mathcal{Q}^+(4n+1, q)$ and let \mathcal{G} be the generator class not containing π . Now, let \mathcal{S} be the set containing π and all generators of \mathcal{G} that

meet π . All elements of $\mathcal{S} \setminus \{\pi\}$ meet π and since all generators of the same class have a nontrivial intersection on this hyperbolic quadric, they also meet each other. Hence, \mathcal{S} is an Erdős-Ko-Rado set. None of the generators in $\mathcal{G} \setminus \mathcal{S}$ extends \mathcal{S} to a larger Erdős-Ko-Rado set. Indeed a generator in $\mathcal{G} \setminus \mathcal{S}$ is disjoint from π . Also, for every generator π' in the same class of π we can find a generator in \mathcal{S} not meeting π' . Consequently, this Erdős-Ko-Rado set is maximal.

The number of generators in \mathcal{S} equals $(|\mathcal{G}| - b_{2n}^{2n}) + 1 = \prod_{i=1}^{2n} (q^i + 1) - q^{2n^2+n} + 1$.

Lemma 3.2. Let \mathcal{S} be a maximal Erdős-Ko-Rado set of generators of a hyperbolic quadric $\mathcal{Q}^+(4n+1, q)$. If \mathcal{S} is not an Erdős-Ko-Rado set as described in Example 1.5 or Example 3.1, then it contains at most $2 \prod_{k=1}^{2n} (q^k + 1) - 2 \min\{W_t^n(q) \mid 1 \leq t \leq n\}$ generators.

Proof. Since \mathcal{S} differs from the Erdős-Ko-Rado sets described in Example 1.5 and Example 3.1, it contains at least two distinct generators of both classes. Let $\pi_1, \pi_2 \in \mathcal{S}$ be two generators of the same class, whose intersection is $2(n-t)$ -dimensional, $t \geq 1$. Then \mathcal{S} contains at most

$$\prod_{k=1}^{2n} (q^k + 1) - W_t^n(q)$$

generators of the other class. The statement immediately follows. \square

Notation 3.3. The function $f_t(q)$, $t \geq 1$, is defined in the following way:

$$f_t(q) := q^{t^2-t} \prod_{k=1}^t (q^{2k-1} - 1) + 2 \sum_{i=0}^{t-1} \begin{bmatrix} 2t \\ 2i \end{bmatrix}_q q^{i^2-i} \prod_{k=1}^i (q^{2k-1} - 1).$$

Using Corollary 2.4, we see that $W_t^n(q) = q^{(n+t)(2n-2t+1)} f_t(q)$.

It should be noted that $f_t(q)$ is independent of n , however closely related to $W_t^n(q)$. We calculate $f_t(q)$ for some small values of t .

$$\begin{aligned} f_1(q) &= q + 1 \\ f_2(q) &= q^6 + q^5 + q^3 - q^2 \\ f_3(q) &= q^{15} + q^{14} + q^{12} - q^{11} + q^{10} - q^9 - q^7 + q^6 \end{aligned}$$

We prove an inequality between these functions.

Lemma 3.4. For every $t \geq 1$ and $q \geq 2$, the inequality $f_{t+1}(q) > q^{4t+1} f_t(q)$ holds.

Proof. We carry out the following calculations.

$$\begin{aligned} f_{t+1}(q) &= q^{(t+1)^2-(t+1)} \prod_{k=1}^{t+1} (q^{2k-1} - 1) + 2 \sum_{i=0}^t \begin{bmatrix} 2t+2 \\ 2i \end{bmatrix}_q q^{i^2-i} \prod_{k=1}^i (q^{2k-1} - 1) \\ &= q^{2t} (q^{2t+1} - 1) \left(q^{t^2-t} \prod_{k=1}^t (q^{2k-1} - 1) \right) + 2 \\ &\quad + 2 \sum_{i=1}^t \frac{(q^{2t+2} - 1)(q^{2t+1} - 1)}{q^{2i} - 1} \begin{bmatrix} 2t \\ 2i-2 \end{bmatrix}_q q^{i^2-i} \prod_{k=1}^{i-1} (q^{2k-1} - 1) \\ &= q^{2t} (q^{2t+1} - 1) \left(q^{t^2-t} \prod_{k=1}^t (q^{2k-1} - 1) \right) + 2 \\ &\quad + 2 \sum_{i=0}^{t-1} \frac{(q^{2t+2} - 1)(q^{2t+1} - 1)q^{2i}}{q^{2i+2} - 1} \begin{bmatrix} 2t \\ 2i \end{bmatrix}_q q^{i^2-i} \prod_{k=1}^i (q^{2k-1} - 1) \end{aligned}$$

Note that

$$\frac{(q^{2t+2} - 1)(q^{2t+1} - 1)q^{2i}}{q^{2i+2} - 1} = q^{4t+1} + \frac{q^{4t+1} - q^{2t+2i+2} - q^{2t+2i+1} + q^{2i}}{q^{2i+2} - 1} > q^{4t+1}$$

since $i \leq t - 1$. Substituting both the equality (for $i = t - 1$) and the inequality in the previous calculation, we find

$$\begin{aligned} f_{t+1}(q) &> q^{2t}(q^{2t+1} - 1) \left(q^{t^2-t} \prod_{k=1}^t (q^{2k-1} - 1) \right) + 2q^{4t+1} \sum_{i=0}^{t-1} \begin{bmatrix} 2t \\ 2i \end{bmatrix}_q q^{i^2-i} \prod_{k=1}^i (q^{2k-1} - 1) \\ &\quad + 2 \frac{q^{4t+1} - q^{4t} - q^{4t-1} + q^{2t-2}}{q^{2t} - 1} \begin{bmatrix} 2t \\ 2t-2 \end{bmatrix}_q q^{(t-1)(t-2)} \prod_{k=1}^{t-1} (q^{2k-1} - 1) \\ &= q^{4t+1} f_t(q) - q^{t^2+t} \prod_{k=1}^t (q^{2k-1} - 1) \\ &\quad + 2 \frac{q^{4t+1} - q^{4t} - q^{4t-1} + q^{2t-2}}{(q^2 - 1)(q - 1)} q^{(t-1)(t-2)} \prod_{k=1}^t (q^{2k-1} - 1) \\ &> q^{4t+1} f_t(q) + (2(q^{4t-2} - q^{4t-4} - q^{4t-5}) - q^{4t-2}) q^{(t-1)(t-2)} \prod_{k=1}^t (q^{2k-1} - 1) \\ &\geq q^{4t+1} f_t(q) . \end{aligned}$$

Here we used that $q^{4t-2} - 2q^{4t-4} - 2q^{4t-5} \geq 0$ for all $q \geq 2$. □

The following inequality was proven in [3].

Lemma 3.5 ([3, Corollary 3.3]). Let $s, t, q \in \mathbb{N}$ be such that $s \leq t$ and $q \geq 3$. If $(s, q) \neq (0, 3)$, then

$$\prod_{i=s}^t (q^i + 1) \leq (q^s + 2) q^{\binom{t+1}{2} - \binom{s+1}{2}} .$$

Corollary 3.6. If $q \geq 3$ and $n \geq 1$, then $q^{2n^2+n} + 2q^{2n^2+n-1} + 1 > \prod_{k=1}^{2n} (q^k + 1)$.

Proof. We apply Lemma 3.5 for $(s, t) = (1, 2n)$ and we find

$$\prod_{i=1}^{2n} (q^i + 1) \leq (q + 2) q^{\binom{2n+1}{2} - 1} - q^{\binom{2n}{2}} < (q + 2) q^{2n^2+n-1} + 1 . \quad \square$$

Now we prove the main theorem of this paper.

Theorem 3.7. The two largest types of maximal Erdős-Ko-Rado sets of generators of a hyperbolic quadric $\mathcal{Q}^+(4n + 1, q)$, $n \geq 1$, $q \geq 3$, are the Erdős-Ko-Rado sets described in Example 1.5 and Example 3.1.

Proof. Let \mathcal{S} be a maximal Erdős-Ko-Rado set of generators different from the Erdős-Ko-Rado sets described in Example 1.5 and Example 3.1. By Lemma 3.2 we know that

$$|\mathcal{S}| \leq 2 \prod_{k=1}^{2n} (q^k + 1) - 2 \min\{W_t^n(q) \mid 1 \leq t \leq n\} .$$

Using Lemma 3.4, we find that

$$W_{t+1}^n(q) = q^{(n+t+1)(2n-2t-1)} f_{t+1}(q) > q^{(n+t)(2n-2t+1)} f_t(q) = W_t^n(q) ,$$

for all $1 \leq t \leq n$. Hence,

$$\min\{W_t^n(q) \mid 1 \leq t \leq n\} = W_1^n(q) = (q+1)q^{2n^2+n-1}.$$

It is clear that the Erdős-Ko-Rado set described in Example 1.5 is larger than the one described in Example 3.1, so we compare the upper bound on $|\mathcal{S}|$ with the size of the Erdős-Ko-Rado set described in Example 3.1. The corresponding inequality

$$\prod_{i=1}^{2n} (q^i + 1) - q^{2n^2+n} + 1 > 2 \prod_{k=1}^{2n} (q^k + 1) - 2(q+1)q^{2n^2+n-1}$$

is equivalent to

$$q^{2n^2+n} + 2q^{2n^2+n-1} + 1 > \prod_{k=1}^{2n} (q^k + 1).$$

By Lemma 3.6 we know that this inequality is valid if $q \geq 3$. \square

Remark 3.8. In the previous theorem the case $q = 2$ is omitted since the key inequality $q^{2n^2+n} + 2q^{2n^2+n-1} + 1 > \prod_{k=1}^{2n} (q^k + 1)$ is not true for $q = 2$ if $n \geq 2$. E.g. $2^{10} + 2 \cdot 2^9 + 1 = 2049 < 2295 = \prod_{k=1}^4 (2^k + 1)$.

For $n = 1$, Theorem 1.7 implies the previous result (also in the case $q = 2$).

4 Some other examples of large Erdős-Ko-Rado sets

Next to the two examples of maximal Erdős-Ko-Rado sets of generators, presented in Example 1.5 and Example 3.1, we also know the *point-pencil*. This is the set of all generators through a fixed point. For many geometries, the point-pencil is the largest Erdős-Ko-Rado set, e.g. for vector spaces ([6, 15]), for designs ([14]), for the polar spaces $\mathcal{Q}^-(2n+1, q)$, $\mathcal{W}(4n+3, q)$, ... ([13]). By Theorem 1.6 we know that this is not true for hyperbolic quadrics $\mathcal{Q}^+(4n+1, q)$. In this case the point-pencil contains

$$\prod_{i=0}^{2n-1} (q^i + 1) = 2 \prod_{i=1}^{2n-1} (q^i + 1) \in \Theta(q^{2n^2-n})$$

generators. Recall that the Erdős-Ko-Rado sets described in Example 1.5 and Example 3.1 contain $\prod_{i=1}^{2n} (q^i + 1) \in \Theta(q^{2n^2+n})$ generators and $\prod_{i=1}^{2n} (q^i + 1) - q^{2n^2+n} + 1 \in \Theta(q^{2n^2+n-1})$ generators, respectively. So, the point-pencil is much smaller in this case. Here we used the Θ -notation: $f \in \Theta(g)$ iff $0 < \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} < \infty$. In other terms, $f = kg + h$, with $k > 0$ a real number and $\lim_{x \rightarrow \infty} \frac{h(x)}{g(x)} = 0$.

In this section we will present some more Erdős-Ko-Rado sets of generators of $\mathcal{Q}^+(4n+1, q)$ whose size is larger than the size of a point-pencil. First we give a counting result.

Lemma 4.1. Let $m \geq 0$ and $k \geq -1$ be two integers such that $k < m$. Let Ω be one of the two generator classes of a hyperbolic quadric $\mathcal{Q}^+(2m+1, q)$. The number of generators in Ω skew to a fixed k -space on the quadric equals

$$\frac{1}{2} \left(\prod_{i=0}^{m-k-1} (q^i + 1) \right) q^{\frac{1}{2}(k+1)(2m-k)} =: w_{m,k}.$$

The empty space is considered to have dimension -1 .

Proof. Let π be a k -dimensional subspace of $\mathcal{Q}^+(2m+1, q)$. We prove this lemma by using induction on k . If $k = -1$, then π is the empty space. The number of generators of Ω skew to the empty space is the total number of generators of Ω , which equals $w_{m,-1}$.

Now, we assume that the lemma is proved for all subspaces of dimension at most $k-1$; we will prove it for a k -dimensional space π . The subspace π contains $\begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_q$ subspaces of dimension i , $0 \leq i \leq k$. Let σ be such an i -space and let $T_\sigma(\mathcal{Q}^+(2m+1, q))$ be its tangent space to $\mathcal{Q}^+(2m+1, q)$. We know that $\mathcal{Q}^+(2m+1, q) \cap T_\sigma(\mathcal{Q}^+(2m+1, q))$ is a cone with vertex σ and base a hyperbolic quadric $\mathcal{Q}^+(2m-2i-1, q)$. The k -space π corresponds to a $(k-i-1)$ -space in this base. Arguing as in the proof of Lemma 2.3, the number of generators of Ω meeting π in precisely σ equals $w_{m-i-1, k-i-1}$. Here, we note that the generators of Ω through σ correspond to one of the two classes of generators of the base $\mathcal{Q}^+(2m-2i-1, q)$.

So, the total number of generators of Ω skew to π is independent of the choice for π , and equals

$$\begin{aligned}
w_{m,k} &= \frac{1}{2} \prod_{j=0}^m (q^j + 1) - \sum_{i=0}^k \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_q w_{m-i-1, k-i-1} \\
&= \frac{1}{2} \prod_{j=0}^m (q^j + 1) - \frac{1}{2} \sum_{i=0}^k \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_q \left(\prod_{j=0}^{m-k-1} (q^j + 1) \right) q^{\frac{1}{2}(k-i)(2m-k-i-1)} \\
&= \frac{1}{2} \left(\prod_{j=0}^{m-k-1} (q^j + 1) \right) \left(\prod_{j=m-k}^m (q^j + 1) - \sum_{i=0}^k \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_q q^{\frac{1}{2}(k-i)(2m-k-i-1)} \right) \\
&= \frac{1}{2} \left(\prod_{j=0}^{m-k-1} (q^j + 1) \right) \left(\prod_{j=m-k}^m (q^j + 1) - \sum_{i=1}^{k+1} \begin{bmatrix} k+1 \\ i \end{bmatrix}_q q^{\frac{1}{2}(k-i+1)(2m-k-i)} \right) \\
&= \frac{1}{2} \left(\prod_{j=0}^{m-k-1} (q^j + 1) \right) \left(\prod_{j=m-k}^m (q^j + 1) - q^{\frac{1}{2}(2m-k)(k+1)} \sum_{i=1}^{k+1} \begin{bmatrix} k+1 \\ i \end{bmatrix}_q q^{\binom{i}{2}} q^{-mi} \right) \\
&= \frac{1}{2} \left(\prod_{j=0}^{m-k-1} (q^j + 1) \right) \left(\prod_{j=m-k}^m (q^j + 1) - q^{\frac{1}{2}(2m-k)(k+1)} \left(\prod_{j=0}^k (q^{j-m} + 1) - 1 \right) \right) \\
&= \frac{1}{2} \left(\prod_{j=0}^{m-k-1} (q^j + 1) \right) q^{\frac{1}{2}(k+1)(2m-k)} .
\end{aligned}$$

In the penultimate transition we used the q -binomial theorem

$$\prod_{l=0}^{n-1} (1 + q^l t) = \sum_{l=0}^n q^{\binom{l}{2}} \begin{bmatrix} n \\ l \end{bmatrix}_q t^l .$$

This calculation finishes the induction step. \square

Remark 4.2. In the previous theorem, the case $m = k$ was not covered; in this case we count the number of generators of a fixed class skew to a given generator π . We already know that this number will be dependent on the class of π and the parity of m . Using the observation before Example 1.5 and Theorem 2.2, we can state the following result. If m is even, then no generators of the class of π are skew to π and $b_m^m = q^{\binom{m+1}{2}}$ generators of the other class are skew to π . If m is odd, then $b_m^m = q^{\binom{m+1}{2}}$ generators of the class of π are skew to π and no generators of the other class are skew to π .

It is an immediate consequence of the previous theorem that the total number of generators skew to a fixed k -space on a hyperbolic quadric $\mathcal{Q}^+(2m+1, q)$, $k < m$, equals

$$\left(\prod_{i=0}^{m-k-1} (q^i + 1) \right) q^{\frac{1}{2}(k+1)(2m-k)} = 2w_{m,k} .$$

We now introduce some new examples of large maximal Erdős-Ko-Rado sets of the hyperbolic quadric $\mathcal{Q}^+(4n+1, q)$.

Example 4.3. Consider the hyperbolic quadric $\mathcal{Q}^+(4n+1, q)$ and let τ be a fixed k -space on it, $0 \leq k \leq 2n$. Denote the two classes of generators by Ω_1 and Ω_2 . Let \mathcal{S} be the union of the set of generators of Ω_1 meeting τ in a subspace of dimension at least j , $0 \leq j \leq k$, and the set of generators of Ω_2 meeting τ in a subspace of dimension at least $k-j$. It is immediate that the elements of \mathcal{S} pairwise intersect. Consequently, \mathcal{S} is an Erdős-Ko-Rado set. We denote this type of Erdős-Ko-Rado sets by $I_{k,j}$. Note that not all these types are different. If $k < 2n$, then $I_{k,j}$ and $I_{k,k-j}$ describe similar sets of generators. Also $I_{2n-1,2j-1}$, $I_{2n,2j-1}$ and $I_{2n,2j}$ describe similar sets of generators, $1 \leq j \leq n$.

We show that an Erdős-Ko-Rado set \mathcal{S} of type $I_{k,j}$ is maximal. Assume that we can find a generator π in $\Omega_1 \setminus \mathcal{S}$ which meets all generators of \mathcal{S} . Since $\pi \notin \mathcal{S}$, we know that $\dim(\pi \cap \tau) < j$. So, we can find a $(k-j)$ -space τ' in τ disjoint to $\pi \cap \tau$. We know that all generators of Ω_2 containing τ' belong to \mathcal{S} . Let $T_{\tau'}$ be the tangent space in τ' to $\mathcal{Q}^+(4n+1, q)$. It is $(4n-k+j)$ -dimensional and meets π in a $(2n-k+j-1)$ -space disjoint to τ' . The intersection $\mathcal{Q}^+(4n+1, q) \cap T_{\tau'}$ is a cone with vertex τ' and base a hyperbolic quadric $\mathcal{Q}_1 := \mathcal{Q}^+(2(2n-k+j-1)+1, q)$. We can choose this basis such that it contains $\pi' = T_{\tau'} \cap \pi$. Moreover $T_{\tau'} \cap \pi$ is a generator of \mathcal{Q}_1 . The set of generators of Ω_2 through τ' , all in \mathcal{S} , correspond to the set of generators of one class of \mathcal{Q}_1 . We denote this class by Ω'_2 . If $k-j$ is even, then π' also belongs to Ω'_2 . However, $2n-k+j-1$ is odd and by an observation made in Remark 4.2, we know that we can find a generator $\sigma \in \Omega'_2$ skew to π' . Then $\langle \pi', \sigma \rangle$ is a generator in \mathcal{S} skew to π . If $k-j$ is odd, then π' belongs to Ω'_1 , the other class of generators of \mathcal{Q}_1 . In this case, $2n-k+j-1$ is even and by an observation made in Remark 4.2, we know that we can find a generator $\sigma \in \Omega'_2$ skew to π' . Then $\langle \pi', \sigma \rangle$ is a generator in \mathcal{S} skew to π . The arguments for $\pi \in \Omega_2 \setminus \mathcal{S}$ are analogous.

Now, we count the number of generators in an Erdős-Ko-Rado set \mathcal{S} of type $I_{k,j}$, $k < 2n$. We use the result from Lemma 4.1.

$$\begin{aligned} |\mathcal{S}| &= \sum_{i=j}^k \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_q w_{2n-i-1, k-i-1} + \sum_{i=k-j}^k \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_q w_{2n-i-1, k-i-1} \\ &= \sum_{i=j}^k \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_q \frac{1}{2} \left(\prod_{p=0}^{2n-k-1} (q^p + 1) \right) q^{\frac{1}{2}(k-i)(4n-k-i-1)} \\ &\quad + \sum_{i=k-j}^k \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_q \frac{1}{2} \left(\prod_{p=0}^{2n-k-1} (q^p + 1) \right) q^{\frac{1}{2}(k-i)(4n-k-i-1)} \\ &= \frac{1}{2} \left(\prod_{p=0}^{2n-k-1} (q^p + 1) \right) \left[\sum_{i=j}^k \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_q q^{\frac{1}{2}(k-i)(4n-k-i-1)} + \sum_{i=k-j}^k \begin{bmatrix} k+1 \\ i+1 \end{bmatrix}_q q^{\frac{1}{2}(k-i)(4n-k-i-1)} \right]. \end{aligned}$$

Using this result, we can see that an Erdős-Ko-Rado set of type $I_{k,j+1}$ is larger than an Erdős-Ko-Rado set of type $I_{k,j}$ if and only if $2j+1-k > 0$, $k < 2n$. Using this and the above mentioned equality between $I_{k,j}$ and $I_{k,k-j}$, we find that the largest among these Erdős-Ko-Rado sets are the ones of type $I_{k,k}$, which are also the ones of type $I_{k,0}$. Those contain

$$\prod_{i=1}^{2n} (q^i + 1) - \frac{1}{2} (q^{\frac{1}{2}(k+1)(4n-k)} - 1) \prod_{i=0}^{2n-k-1} (q^i + 1) \in \Theta(q^{2n^2-n+k})$$

generators. In this computation we used the q -binomial theorem. Since the Erdős-Ko-Rado sets of type $I_{2n,2j-1}$ and $I_{2n,2j}$ are equal to the Erdős-Ko-Rado sets $I_{2n-1,2j-1}$, $1 \leq j \leq n$, we can use the above formulas to compute their number of elements as well.

So, the only type $I_{k,j}$ of Erdős-Ko-Rado sets whose size has not been computed above is $I_{2n,0}$. However, the Erdős-Ko-Rado sets of type $I_{2n,0}$ are precisely the ones that are described in

Example 1.5. Furthermore, the Erdős-Ko-Rado sets of type $I_{2n,2n}$ are the ones that are described in Example 3.1 and the Erdős-Ko-Rado sets of type $I_{0,0}$ are the point-pencils.

In the previous example we have introduced several types of Erdős-Ko-Rado sets, $I_{k,k}$, whose size is larger than the size of a point-pencil. These are however not the only ones. We shall give two more examples.

Example 4.4. Again we denote the two classes of generators on $\mathcal{Q}^+(4n+1, q)$ by Ω_1 and Ω_2 . Let π be a generator of class Ω_1 and let τ be a fixed k -space in π , $0 \leq k \leq 2n-2$. Let \mathcal{S} be the union of the set of generators of Ω_1 that are not skew to τ or that meet π in a subspace of dimension $i \geq 2$, and the set of generators of Ω_2 through τ meeting π in a subspace of dimension $2n-1$. It is immediate that the generators in \mathcal{S} pairwise intersect, and hence \mathcal{S} is an Erdős-Ko-Rado set. We denote this type of Erdős-Ko-Rado sets by II_k . Its maximality can be proven by arguments similar to the arguments in the proof of the maximality in Example 4.3.

In the above definition we imposed $k \leq 2n-2$. For $k = 2n-1$ this definition gives rise to the Erdős-Ko-Rado set described in Example 3.1; for $k = 2n$ this definition gives rise to the Erdős-Ko-Rado set described in Example 1.5.

We now count the number of generators in an Erdős-Ko-Rado set \mathcal{S} of type II_k :

$$\begin{aligned} |\mathcal{S}| &= \sum_{i=1}^n \begin{bmatrix} 2n+1 \\ 2i+1 \end{bmatrix}_q b_{2n-2i-1}^{2n-2i-1} + \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q b_{2n-1}^{2n-1} + \begin{bmatrix} 2n-k \\ 1 \end{bmatrix}_q \\ &= \sum_{i=1}^n \begin{bmatrix} 2n+1 \\ 2i+1 \end{bmatrix}_q q^{\binom{2(n-i)}{2}} + \begin{bmatrix} k+1 \\ 1 \end{bmatrix}_q q^{2n^2-n} + \begin{bmatrix} 2n-k \\ 1 \end{bmatrix}_q. \end{aligned}$$

It can be calculated that an Erdős-Ko-Rado set of type II_k contains more elements than an Erdős-Ko-Rado set \mathcal{S} of type $II_{k'}$ if and only if $k > k'$. Therefore, we calculate the size of an Erdős-Ko-Rado set \mathcal{S} of type II_{2n-2} :

$$\begin{aligned} |\mathcal{S}| &= \sum_{i=1}^n \begin{bmatrix} 2n+1 \\ 2i+1 \end{bmatrix}_q q^{\binom{2(n-i)}{2}} + \begin{bmatrix} 2n-1 \\ 1 \end{bmatrix}_q q^{2n^2-n} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \\ &= \sum_{i=1}^{n-1} q^{\binom{2(n-i)}{2}} \left(q^{2(n-i)} \begin{bmatrix} 2n \\ 2n-2i \end{bmatrix}_q + \begin{bmatrix} 2n \\ 2n-2i-1 \end{bmatrix}_q \right) + 1 + \frac{q^{2n-1}-1}{q-1} q^{2n^2-n} + q + 1 \\ &= \sum_{j=0}^{2n-2} \begin{bmatrix} 2n \\ j \end{bmatrix}_q q^{\binom{j}{2}} q^j + \frac{q^{2n-1}-1}{q-1} q^{2n^2-n} + q + 1 \\ &= \sum_{j=0}^{2n} \begin{bmatrix} 2n \\ j \end{bmatrix}_q q^{\binom{j}{2}} q^j - q^{2n^2+n} - \frac{q^{2n}-1}{q-1} q^{2n^2-n} + \frac{q^{2n-1}-1}{q-1} q^{2n^2-n} + q + 1 \\ &= \prod_{i=1}^{2n} (q^i + 1) - q^{2n^2+n} - q^{2n^2+n-1} + q + 1 \in \Theta(q^{2n^2+n-2}). \end{aligned}$$

Analogously, the size of an Erdős-Ko-Rado set \mathcal{S} of type II_0 can be calculated. We find

$$|\mathcal{S}| = \prod_{i=1}^{2n} (q^i + 1) - \frac{(q^{2n}-1)(q^{2n^2-n+1}-1)}{q-1} \in \Theta(q^{2n^2+n-3}).$$

Example 4.5. Before introducing the example, we recall the *triatlity* map for $\mathcal{Q}^+(7, q)$, which has its origins in [16]; we follow the approach from [11]. Denote the two generator classes of $\mathcal{Q}^+(7, q)$ by Ω'_1 and Ω'_2 . Let \mathcal{P} be the set of points on $\mathcal{Q}^+(7, q)$ and let \mathcal{L} be the set of lines on $\mathcal{Q}^+(7, q)$. Note that $|\mathcal{P}| = |\Omega'_1| = |\Omega'_2|$. A D_4 -geometry \mathcal{G} can be constructed as follows. The elements of \mathcal{P} are the 0-points, the elements of Ω_i are the i -points, $i = 1, 2$, and the elements of \mathcal{L} are the lines.

Incidence is defined by symmetrized containment for all combinations of two groups, except for 1-points and 2-points; we define a 1-point and a 2-point to be incident if they meet in a plane of $\mathcal{Q}^+(7, q)$. Every permutation of $\{\mathcal{P}, \Omega'_1, \Omega'_2\}$ defines a geometry isomorphic to \mathcal{G} . A triality of \mathcal{G} is a map t

$$t : \mathcal{L} \rightarrow \mathcal{L}, \mathcal{P} \rightarrow \Omega'_1, \Omega'_1 \rightarrow \Omega'_2, \Omega'_2 \rightarrow \mathcal{P}$$

preserving the incidence in \mathcal{G} and such that t^3 is the identity relation. Such maps are known to exist and are used to construct generalised hexagons.

We use the triality to prove a result about the generators of $\mathcal{Q}^+(7, q)$. Let π_1 and π_2 be two disjoint generators of Ω'_2 and let \mathcal{S}' be the set of all generators of Ω'_2 meeting both π_1 and π_2 in a line. Let \mathcal{S}'' be the set of all generators of Ω'_2 having a nonempty intersection with all elements of \mathcal{S}' . We will show that $\mathcal{S}'' = \{\pi_1, \pi_2\}$. It is clear that π_1^t and π_2^t are two points not on a line of \mathcal{L} . The set \mathcal{S}^t contains all points of $\mathcal{Q}^+(7, q)$ that are collinear with both π_1^t and π_2^t . Therefore, \mathcal{S}^t is the set of points on a hyperbolic quadric $\mathcal{Q}^+(5, q)$ inside $\mathcal{Q}^+(7, q)$. The only points collinear with all points of \mathcal{S}^t are the points π_1^t and π_2^t themselves. Hence, $\mathcal{S}''^t = \{\pi_1^t, \pi_2^t\}$. The statement follows. Note that we can replace Ω'_2 by Ω'_1 in the statement; replacing t by t^2 , this proof continues.

Now, we consider the hyperbolic quadric $\mathcal{Q}^+(4n+1, q)$, $n \geq 2$. Denote the two classes of generators by Ω_1 and Ω_2 . Let π and π' be two generators of Ω_1 meeting in a $(2n-4)$ -space τ . Let \mathcal{S} be the set containing π , π' and all generators of Ω_2 meeting π and π' . It is clear that \mathcal{S} is an Erdős-Ko-Rado set. We denote this type of Erdős-Ko-Rado sets by *III*.

We prove that an Erdős-Ko-Rado set \mathcal{S} of type *III* is maximal. It is obvious that no generators of Ω_2 that are not in \mathcal{S} extend \mathcal{S} . Let π'' be a generator of Ω_1 meeting all generators of \mathcal{S} . We note that \mathcal{S} contains all generators of Ω_2 having a nonempty intersection with τ . We show that π'' has to contain τ . If π'' does not contain τ , then we can find a point $P \in \tau \setminus \pi''$. Through P we can find a generator $\sigma' \in \Omega_2$ skew to π'' (by Remark 4.2 there are b_{2n-1}^{2n-1} such generators). This is a contradiction since σ' clearly is contained in \mathcal{S} . Hence, we know that π'' contains τ . Now we consider the tangent space T_τ in τ to $\mathcal{Q}^+(4n+1, q)$. The intersection $T_\tau \cap \mathcal{Q}^+(4n+1, q)$ is a cone with vertex τ and base a hyperbolic quadric $\mathcal{Q}^+(7, q)$. The generators π , π' and π'' intersect this base in $\bar{\pi}$, $\bar{\pi}'$ and $\bar{\pi}''$, respectively. These are generators of the hyperbolic quadric $\mathcal{Q}^+(7, q)$ of the same class, say Ω'_2 . Let σ be a generator of $\mathcal{Q}^+(7, q)$ of class Ω'_2 , meeting both $\bar{\pi}$ and $\bar{\pi}'$ in a line. We know that there are b_{2n-4}^{2n-4} generators of $\mathcal{Q}^+(4n+1, q)$ through σ disjoint to τ , necessarily all of class Ω_2 ; these generators are contained in \mathcal{S} . Hence $\bar{\pi}''$ has to meet all generators of $\mathcal{Q}^+(7, q)$ of class Ω'_2 that meet both $\bar{\pi}$ and $\bar{\pi}'$ in a line, as σ could be arbitrarily chosen. By the above observation on $\mathcal{Q}^+(7, q)$, we know that $\bar{\pi}'' \in \{\bar{\pi}, \bar{\pi}'\}$. Hence $\pi'' \in \{\pi, \pi'\}$, and consequently \mathcal{S} is maximal.

We count the number of generators in an Erdős-Ko-Rado set \mathcal{S} of type *III*.

$$\begin{aligned} |\mathcal{S}| &= 2 + \left(\prod_{i=1}^{2n} (q^i + 1) - w_{2n, 2n-4} \right) + \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q q^{4(2n-3)} b_{2n-4}^{2n-4} \\ &= 2 + \prod_{i=1}^{2n} (q^i + 1) - \left(\prod_{i=1}^3 (q^i + 1) \right) q^{(n+2)(2n-3)} + (q^2 + 1)(q^2 + q + 1)q^{(n+2)(2n-3)} \\ &= \prod_{i=1}^{2n} (q^i + 1) - q^{2n^2+n-6} (q^6 + q^5 + q^3 - q^2) + 2 \in \Theta(q^{2n^2+n-2}) \end{aligned}$$

Remark 4.6. We already noted that the size of the largest maximal Erdős-Ko-Rado set is of order $\Theta(q^{2n^2+n})$ and the size of the second largest maximal Erdős-Ko-Rado set is of order $\Theta(q^{2n^2+n-1})$. In the previous Examples we have described three types of maximal Erdős-Ko-Rado sets of the next order $\Theta(q^{2n^2+n-2})$, namely $I_{2n-2, 2n-2}$, II_{2n-2} and *III*. However, it should be noted that the Erdős-Ko-Rado sets of type $I_{2n-2, 2n-2}$ and Erdős-Ko-Rado sets of type II_{2n-2} are the same ones. This is an exceptional case; this pattern does not continue for other Erdős-Ko-Rado sets of type $I_{k,k}$ and $II_{k'}$. It can be easily calculated that the Erdős-Ko-Rado sets of type $I_{2n-2, 2n-2}$ (II_{2n-2}) are larger than the Erdős-Ko-Rado sets of type *III*.

Calculations in Example 4.3 and Example 4.4 show that there are many different Erdős-Ko-Rado sets whose size is larger than the size of the point-pencil. So, a complete classification of all Erdős-Ko-Rado sets whose size is at least the size of a point-pencil is out of sight for the moment.

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